OBJECTIVE :

To prove that all axioms (except projections, were a modified version of projections is proved here) of the finite axiomatization of NFU present in the online article : Elementary set theory with a universal set. are proved by CST.

EXPOSITION OF CONNECTION SET THEORY "CST"

CST is the set of all sentences entailed (from FOL with identity "=", and membership "e") by the axiom scheme outlined below the following definition:

Define(set):- x is a set iff ~ Exist y (for all z (z e y iff z e x) and ~ y=x)

Comprehension: If phi is a well connected formula in which y is free, and in which x do not occur, then all closures of

Exist a set x for all y (y e x iff phi)

are axioms.

Definition of well connected formulas:

We say that a variable x is connected to a variable y in phi iff any of the following formulas appear in phi

x e y , y e x , y=x , x=y

We refer to a function s from {1,...,n} to variables in phi as a chain of length n in phi iff for each appropriate index i: s(i) is connected to s(i+1), and s(i),s(i+2) are two different occurrences in phi(i.e. s(i) occur at a place in phi that is different from the place s(i+2) occur at).

A chain from x to y is defined as a chain s of length n>1 with s(1)=x and s(n)=y.

phi is a well connected formula iff for each variable x in phi there is no chain from x to x.

/ Theory definition finished.

I shall use the term "connected formula" for short to refer to "well connected formula".

EXTENSIONALITY

The formula “y e a” is connected!

Then for every object “a” we have the set x ={y| yea}
From the definition of sets there is no object other than x that has exactly the same members as x, then “a” cannot be a “non set” since by then it would be a different object from x (identity theory), that has the same members as x, which contradict the set-hood of x, and since from comprehension “every” object “a” will have a set x that has exactly all members of a, then every object is a set!

UNIVERSE

“Exist s (s e y) OR ~ Exist r (r e y)” is a connected formula!

Substitute in comprehension and we get the set of all objects “V”.

COMPLEMENTS

“~ y e a” is a connected formula!

Substitute in comprehension and we get x= {y| ~ y e a}
So x is the absolute complementary set of a.

BOOLEAN UNION

“ y e a OR y e b” is a connected formula!

Substitute in comprehension and we get x={y| y e a or y e b}
Which is the Boolean union of a and b.

SET UNION

“Exist z (z e a & y e z)” is a connected formula!

Substitute in comprehension and we get x=Union a.

SINGLETONS

“y=a” is a connected formula.

Substitute in comprehension and we get x={a}.

ORDERED PAIRS

Ordered pairs *are* definable in CST.

Wiener's Pairs work!

x=<a,b>= {{{a},0},{{b}}}

Exist a,b (for all y (y e x <-> (for all w (w e y <-> (for all k (k e w <-> k=a) or
not Exist s (s e w)))
or

for all u (u e y <-> for all n (n e u <-> n=b))))).

The above formula is a connected formula!

Let’s call the above formula as:

“x is a Wiener ordered pair from a to b”

Note: Whenever the term “ordered pair” is mentioned in this account, it refers to Wiener’s ordered pair, so the notation <a,b> refers to Wiener’s ordered pair from a to b.

THE CARTESIAN PRODUCT

“ x is a Wiener ordered pair from a to b & a e A & b e B”

is a connected formula!

Substitute in comprehension and we get

A×B ={x| x is a Wiener ordered pair from a to b & a e A & b e B}.

Or simply

A×B={<a,b>| a e A & b e B}.

A relation is defined as a member of the powerset of the set of all ordered pairs.

It should be noted that Power and Pairing are theorems of CST.

THE DOMAIN OF A RELATION

For every relation R, the following formula is connected!

"Exist m e R Exist pq (y e p , p e q , 0 e q , q e m)"

The above formula states that there exists a member "m" of R such that y is the first projection of m (notice that "0 e q" forces y to be the first projection of m)

Substitute in comprehension and we get Dom(R).

The same above formula but with ~0 e q, would yield the Range of R.

THE CONVERSE OF A RELATION

The following formula is connected!

Let Phi(y,R)<->

(y is a Wiener ordered pair &

Exist meR For all i Exist k,l,p,q (((iek,kel,ley)<->(iep,peq,qem))

& (~0el <-> 0eq))).

So: {y|Phi(y,R)}={<a,b>| <b,a> e R}

Note: to avoid confusion, please review the end of this document to see exactly how to treat variables linked to the symbol "0".

SINGLETON IMAGES

Let Phi(y,R)<->

(y is a Wiener ordered pair

&

Exist meR For all i Exist k,l,p,q ((0el <-> 0eq) &
((iek,kel,ley)<-> Exist j (jep,peq,qem & for all z (zei<->z=j))))).

So: {y|Phi(y,R)}={<{a},{b}>|<a,b> e R}

Note: to avoid confusion, please review the end of this document to see exactly how to treat variables linked to the symbol "0".

RELATIVE PRODUCT OF RELATIONS

We want to prove that for every relations R and S the set R|S named as the relative product of R and S exists

R|S={<a,b>| Exist c(<a,c> e R & <c,b> e S)}

Now to prove that, first we define the following set C

C= {<p,q>| p e R , q e S}

this can be defined in this theory, were <p,q> is a Wiener ordered pair from p to q, notice that C=R×S.

Now we use the following formula to construct the set C"

Phi(y,C) <->
y e C & Exist!i Exist f,g,h,k,l (ief,feg,geh,hek,kel,ley &(~0eg<->0el))

Now the set C"={y|Phi(y,C)} would be equal to the set

{<<a,c>,<c,b>>| <a,c> e R , <c,b> e S}

Now it is easy to obtain R|S from C".

Let Phi(y,C")<-> (y is a Wiener ordered pair
&
Exist meC" For all i Exist f,g,h,k,l,w,u
((ief,feg,geh,hek,kel,lem <->iew,weu,uey)&
((0eu,0el,0eg) or (~0eu,~0el,~0eg)))).

So: {y|Phi(y,C")}=R|S

<u>Note:</u> to avoid confusion, please review the end of this document to see exactly how to treat variables linked to the symbol “0”.

THE DIAGONAL SET

Phi(y) <-> (y is a Wiener ordered pair &
Exist! i Exist pq (iep,peq,qey)).

Substitute in comprehension and we get x= {y| Phi(y)} ={<i,i>| i e V}

QED.

SUBSET RELATION

The following formula is connected!

"for all y (y e t <-> for all w(
Exist k (for all u (u e w <-> (u is a Wiener ordered pair &
for all i (Exist sr (ies,ser,reu) -> i subset k) &
for all j (Exist pq (jep,peq,0eq,qeu) -> j=x))))

-> y e w))".

Now for each set x, any set w is a set of all ordered pairs
having x as their first projection and in which both projections in each pair in w are subsets of some set k, were k is of course a superset of x.

Now for each set x, we will have many sets w each defined
after a set k, now we can see that for each set x there can exist a minimal set of all w sets, and this minimal happens when k=x, because k is a superset of x, then all pairs of the form <x,a> were a is a subset of x (i.e. k=x) would be present in any set w for any k because k superset x and because any subset of x would be a subset of k, and since we have intersection as a theorem of CST, then we can have the minimal of all w sets for each set x, which would be the intersection of all sets w for each set x, which is the set t above.

So t will be the set of all ordered pairs <x,a> were a is a subset of x.

Now for each x we will have a set t.

The above formula is a connected formula, so we can add "Exist x" to the above formula to get a connected formula also,

"Exist x (
for all y (y e t <-> for all w(
Exist k (for all u (u e w <-> (u is a Wiener ordered pair &
for all i (Exist sr (ies,ser,reu) -> i subset k) &
for all j (Exist pq (jep,peq,0eq,qeu) -> j=x))))

-> y e w)))".

Substitute in comprehension and we'll get the set G; note the set builder variable is t,
so comprehension reads

Exist a set G for all t (t e G <-> "the above formula")

Now Union G would be the set coding the subset relation.

QED.

PROJECTIONS

In CST we can have the following projection sets

{<<x,y>,{{{x}}}> | x,y e V} and {<<x,y>,{{{y}}}> | x,y e V}

(where <x,y> = {{{x},0},{{y}}} which is Wiener's pairs).

Wiener's pairs are one level higher than Kuratowski's
that's why we need the extra bracket over the projections.

The problem with Kuratowski pairs is that it cannot be
constructed using connected comprehension, while Wiener's pairs are.

Proof:

First we construct the set V1 which is the set of all Wiener ordered pairs
having their first projection as a Wiener ordered pair and their
Second projection as a singleton set of a singleton set of a singleton set.

then we construct V2 which is the subset of V1 having maximally *two*
"members of members of members "up to six times iteration" " of each y in V1.

V2 would be the set of all pairs <<x,y>,{{{x}}}> and pairs
<<x,y>,{{{y}}}>.

then we construct V3 which is the subset of V2 having unique
member of a member of a member (0 not in it) of a member (complete it up to six times)
... of each y in V2

V3 would be the set of all Wiener pairs <<x,y>,{{{y}}}>

Now we also can construct a subset of V1 having maximally ONE member of member of
member(up to six times)..of each y in V1, and this set is V!, which is the set of all pairs of
the form <<x,x>,{{{x}}}>.

Now we construct the set V4 which is V2\V3 Union V!
and V4 is the set of all Wiener pairs <<x,y>,{{{x}}}>.

So V3 and V4 are the needed projections.

QED.

IMPRESSION: It appears that CST is NF itself!

Projections as defined here, parallel those defined after Kuratowski ordered pairs in some versions of the finite axiomatization of NF; thus Stratification would be proved in CST, and by then NF would be a sub-theory of CST, and since all connected formulas are stratified, and CST is a sub-theory of NF, then:

CST=NF.

<u>TO AVOID CONFUSION:</u>

The formula 0eq <-> 0el is meant to be written as:

Exist k (k e q & ~ Exist s (s e k)) <->
Exist d (d e l & ~ Exist m (m e d)).

So as one can see, there is no chain from q to l, nor from l to q,
The above formula is a well connected formula.

Also the formula

0eq,0el,0eu

is to be written as

Exist k (k e q & ~ Exist s (s e k)) and
Exist d (d e l & ~ Exist m (m e d)) and
Exist j (j e u & ~ Exist n (n e j)).

So as one can see, there are no chains between the variables q,l,u.

So the main point is that the notation using the symbol 0 in them is just written for convenience, it is actually a shorthand for the more extensive notation that render all variables linked by the symbol 0 as not linked by any chain from one to another.

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