Definitional Typed Second Order Logic

This is an extension of first order logic with identity in a *Fregean* manner; it adds quantification over predicate symbols, and stipulate object extensions for typed predicates. Typed predicates are defined recursively after rules of formation of typed formulas. It constitutes a possible salvage for Frege's project of reducing mathematics to logic.

Notation:

ε is a monadic symbol denoting "extension of"

Object symbols shall be denoted by lower case.

Predicate symbols shall be denoted in upper case

Typed predicates shall be denoted by indexed predicate symbols.

Straight predicate symbols represent Constant predicates

Italic predicate symbols represent Variables ranging over Constant predicates of the same index.

for example: ***P1*** is a variable symbol ranging over all Constant predicates indexed with 1, so it ranges over **Q1,P1,R1,....,** so it can only be substituted by those.

While **P1** represents a particular predicate.

All first order with identity logic formulas has all predicate symbols in them being constant predicate symbols.

Formation rules of typed formulas:

Rule 0: Any predicate in a first order formula is a typed constant predicate symbol.

Rule 1: if ***Pi*** is a typed predicate symbol then ε***Pi*** is a term.

Rule 2: Any first order with identity logic formula if we index all predicates (except =) in it with 1 then the resulting formula is a typed formula.

Example: ∀x. **P1**(x) → x=ε**P1**

Rule 3: *Italicing* predicate symbols in a typed formula results in a typed formula.

Example: ∀x. ***P1***(x) → x=ε***P1***

is a typed formula.

Rule 4: quantifying over variable predicates of a typed formula results in a typed formula

so "∃***P1***. ∀x. ***P1***(x) → x=ε***P1***" is a typed formula.

Rule 5: If a formula **F** is a definitional formula of predicate **Q** after a typed formula **Gn** (**Gn** has the highest index of a predicate in it being **n**), and if all of those highest indexed predicates were constant predicates and if **Q** received the same index **n**, then **F** is a typed formula.

In general **F** is a definitional formula of predicate **Q** after formula **G** means **F** is a formula of the form "∀x. **Q**(x) ↔ **G**".

Rule 6: For the same conditions in Rule 5, if any of the highest indexed predicates in **Gn** is a variable predicate symbol, then **Q** must receive index **n+1** in order for **F** to be a typed formula.

Examples:

$\forall x.\ \mathbf{Qi+1}(x) \leftrightarrow \neg\exists \boldsymbol{Pi}.\ \boldsymbol{Pi}(\varepsilon \boldsymbol{Pi}) \wedge x=\varepsilon \boldsymbol{Pi}$

$\forall x.\ \mathbf{Qi}(x) \leftrightarrow \mathbf{Pi}(x) \wedge \neg \mathbf{Gi}(x)$

are typed formulas.

Rule 7: a typed predicate symbol (any predicate symbol in a typed formula) only range over predicates that hold of OBJECTS only.

Rule 8: if a formula is a typed formula, then ALL of its sub-formulas are typed!

Rule 9: if **P**,**Q** are typed formulas, then **P**|**Q** is a typed formula; where "|" is the Sheffer stroke.

Rule 10: all propositional logic equivalents of any typed formula are typed formulas.

So for example: " $\forall x.\ \neg\ [\mathbf{Qi+1}(x) \oplus (\neg\exists \boldsymbol{Pi}.\ \boldsymbol{Pi}(\varepsilon \boldsymbol{Pi}) \wedge x=\varepsilon \boldsymbol{Pi})]$" is a typed formula.

Now the above rules will recursively form typed formulas, and typed predicates.

Axiom: if ***Pi***,***Qj*** are typed predicates, then:

$\varepsilon \boldsymbol{Pi}=\varepsilon \boldsymbol{Qj}$ iff $(\forall x.\ \boldsymbol{Pi}(x) \leftrightarrow \boldsymbol{Qj}(x))$

Define: $x \in y$ iff $\exists \boldsymbol{G}.\ \boldsymbol{G}(x) \wedge y=\varepsilon \boldsymbol{G}$

The motivation beyond extending predicates is to reduce object/predicate/higher predicate hierarchy into object/predicate dichotomy, thus enabling Rule 7.

The above system have no mathematical motivation, it is solely derived by consistency concerns about second order logic with identity using purely logically motivated maxims.

The above ***LOGIC*** have the ability to interpret ***second order arithmetic***, thus most of traditional mathematics can be seen to be traced to logic, thereby highly motivating ***Logicism.***

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