Short Axiomtization of Stratified Comprehension

By Zuhair Al-Johar

18/9/2014

Claim: SF is finitely axiomatized in the language of set theory (i.e., FOL($=$, $\in$)) by:

[1] Sheffer strokes: $\forall A,B\ \exists X\ \forall y\ (y \in X \Leftrightarrow \neg(y \in A \land y \in B))$

[2] Singletons: $\forall A\ \exists X\ \forall y\ (y \in X \Leftrightarrow y = A)$

[3] Set Unions: $\forall A\ \exists X\ \forall y\ (y \in X \Leftrightarrow \exists k\ (k \in A \land y \in k))$

Define: $pair(x,a,b) \Leftrightarrow \forall y\ (y \in x \Leftrightarrow y=a \lor y=b)$

[4] Unordered Relative Products (Composition):
$\forall R,S\ \exists X\ \forall y\ (y \in X \Leftrightarrow \exists a,b,c,r,s\ (pair(y,a,c) \land r \in R \land s \in S \land pair(r,a,b) \land pair(s,b,c)))$

[5] Unordered Intersection relation set:
$\exists X\ \forall y\ (y \in X \Leftrightarrow \exists a,b,c\ (pair(y,a,b) \land c \in a \land c \in b))$
/

Notation: the unordered relative product will be denoted by the colon : ; an unordered intersection relation set is denoted by $\{\cap\}$. All monadic symbols of complements, set union and powers, etc... are to take PRECEDENCE over dyadic symbols, so $x \cap y'$ means an intersection set of x with a complementary set of y. For purposes of clarity we resort to using function symbols to witness existence of sets in order to accommodate for absence of Extensionality. Alternatively one can write all functions as predicates (relations), like in writing $\cup(X,A,B)$ to signify X is a Boolean union of A and B.

Proof: We proceed to prove that each of the 14 point axiomatic system, present in Holmes's online article "*Elementary set theory with a universal set*", is provable here:

$A' = \{x|\neg(x \in A \land x \in A)\}$; $V = \{x|\neg(x \in A \land x \in A')\}$; $A \cup B=\{x|\neg(x \in A' \land x \in B')\}$;
$A \cap B = (A' \cup B')'$; $A\backslash B = A \cap B'$; $\{A\} = \{x|\ x=A\}$; $\{A,B\} = \{A\} \cup \{B\}$;
$\cup(A) = \{x|\ \exists k\ (k \in A \land x \in k)\}$

Lemma of Singleton Power: For every set A there exists a set of all singleton subsets of A.

Proof: That power holds for every set is easily proven. Take any set A, take one of its complements A', now take $(\{\{A'\}\}:(V:V)) \cap \{\cap\}$ and take a union of it and we get a set of all sets intersecting with A'. Now a complement of that set is P(A).

Now V:V is a set of all pairs, call it 2*. We proceed to filter out doubletons from 2* in order to construct a Frege numeral 1. The basic idea of the proof is that singletons differ from doubletons in not being able to intersect with two disjoint sets, while all doubletons can! The proof proceeds as follows:

Take $P(2^*) \cap 2^*$, this is a set of all pairs of pairs call it H. Now take H:V and we get a set of all pairs {{a,b},x}. Let G be $P((2^*)') \cap 2^*$, which is a set of all pairs of non pairs; take G:V and we get a set of all pairs {k,x} where k is not a pair. Now take $(G{:}V) \cap (H{:}V)$ and we get a set of all pairs {k,{a,b}} where k is not a pair, call it I. Also take a set $G\backslash\{\cap\}$ and this would be a set of all pairs {m,n} where neither m nor n is a pair, and where m and n are disjoint! Call it J.

Now: $2 = \cup((J{:}(I \cap \{\cap\})) \cap \{\cap\}) \cap 2^*$
Accordingly: $1 = 2^*\backslash 2$

A singleton power of "any" object X, denoted as "X^{ι}", is: $P(X) \cap 1$.

Define: $X^{\iota 2} = (X^{\iota})^{\iota}$

Proof of Diagonal: $[=] = 1^{\iota}$

Proof of Domains: $dom(R) = \cup(\cup(R) \cap 1)$

Lemma of unordered Products: $\forall A,B\ (\{\{a,b\} \mid a \in A \wedge b \in B\}$ exists)
Define: $A^*B = \{\{a,b\} \mid a \in A \wedge b \in B\}$

Proof: $A^*B = (A^{\iota}{:}V) \cap (B^{\iota}{:}V)$

Proof of Cartesian products: $AxB = (A^{\iota} * (A^*B)) \cap \{\cap\}$
Proof of Intersection relation set: $[\cap] = (1 * \{\cap\}) \cap \{\cap\}$

Proof of 1st projection relation set: $\pi_1 = (1^{\iota}\ x\ (VxV)) \cap [\cap]$
Frege numeral zero can be easily constructed as: $0 = (\cup(\{\cap\}))'$

Proof of $\{\cap^1\}$: $\{\cap^1\}$ is:

$(((((V\backslash(0 \cup 1))^*1^{\iota}) \cap \{\cap\}) : (((V\backslash(0 \cup 1))^*1^{\iota}) \cap \{\cap\})) \cap L) \cup (((V\backslash(0 \cup 1))^*1^{\iota}) \cap \{\cap\}) \ \cup 1^{\iota 2}$

$\{\cap^1\}$ is a set of all pairs {x,y} where x intersect y and an intersection set of x and y contains at least one element that is a singleton set.

Where $L = \cup((1')^{\iota 2}{:}\{\in^*\})$; for L is a set of all sets in which there is an element that is not a singleton. $\{\in^*\} = (1^*V) \cap \{\cap\}$

Lemma of Relative Products: $\forall R,S\ (\{(a,c) \mid \exists b\ ((a,b) \in R \wedge (b,c) \in S)\}$ exists)
where (,) is the Kuratowski implementation of ordered pair.
Definition: $R|S = \{(a,c) \mid \exists b\ ((a,b) \in R \wedge (b,c) \in S)\}$

Proof: The idea is to construct a set R|S, in stages as follows:
We divide each of R and S sets into three disjoint subsets R_1, R_2, R_3 and S_1, S_2, S_3; now R_1 and S_1 shall denote the sets of all identical pairs in R and S respectively, and

since we are implementing the Kuratowski construction for ordered pairs, then those would be the sets of all singleton elements of R and S respectively. R_2 is meant to be a set of ALL elements of R that have non singleton converses in S, and so S_2 is a set of ALL elements of S that have non singleton converses in R. What remains are sets R_3, S_3 which are subsets of R and S that also have all their elements being Doubletons (i.e. non singleton ordered pairs), but such that for each $(a,b) \in R_3$ there do NOT exist $(b,a) \in S$, and also for each $(a,b) \in S_3$ there do NOT exist $(b,a) \in R$.

Now the main line of the proof is that a set $R|S$ would be a union of relative products at each individual level, formally speaking this is:

$R|S= (R_1|S_1) \cup (R_1|S_2) \cup (R_1|S_3) \cup (R_2|S_1) \cup (R_2|S_2) \cup (R_2|S_3) \cup (R_3|S_1) \cup (R_3|S_2)$
$\cup (R_3|S_3)$

However due to specifics of the proof, that would be shortened to the following five stages: $R|S= (R_1|S) \cup ((R_2 \cup R_3)|S_1) \cup (R_2|S_2) \cup (R_3|(S_2 \cup S_3)) \cup (R_2|S_3)$

Now we build up each of the above stage sets:

First we define each of the above segments of R and S.
$R_1= R \cap 1$
$S_1= S \cap 1$
$R_2= (R \backslash R_1) \cap \cup((R*S) \cap (\{\cap\}\backslash\{\cap^1\}))$
$S_2= (S \backslash S_1) \cap \cup((R*S) \cap (\{\cap\}\backslash\{\cap^1\}))$
$R_3= R\backslash(R_1 \cup R_2)$
$S_3= S\backslash(S_1 \cup S_2)$

Second we come to build the stage relative product sets:
(1) $R_1|S = R_1{:}S$
(2) $(R_2 \cup R_3)|S_1 = (((\cup(R_2 \cup R_3) \cap 2) : \cup(S_1)) * 1) \cap (R_2 \cup R_3)$
(3) $R_2|S_2 = (\cup(R_2) \cap 1)^{\iota}$
(4) $(R_3|(S_2 \cup S_3)) = (A^*{:}B^*) \cap \{\cap\}$
where $A^*= (R_3{:}((2*2) \cap \{\cap\}))\backslash\{\cap\}$
This is a set of all elements of the form $\{\{a\},\{b,x\}\}$ where $x \neq a$, $x \neq b$; and where $(a,b) \in R_3$.
Let $B = ((S_2 \cup S_3){:}((2*2) \cap \{\cap\}))\backslash\{\cap\}$
This is a set of all elements of the form $\{\{b\},\{c,y\}\}$ where $y \neq b$, $c \neq y$; and where $(b,c) \in (S_2 \cup S_3)$
Now: $B^*= (((1*2) \cap \{\cap\}){:}B)\backslash(1*1)$
So B^* is a set of all elements of the form $\{\{b,z\},\{c,y\}\}$, where $(b,c) \in (S_2 \cup S_3)$, and $y \neq b$, $c \neq y$, $b \neq z$.
(5) $(R_2|S_3) = (A^*{:}B^*) \cap \{\cap\}$
Where A^*, B^* are constructed after the same steps in (4) but with R_2 replacing R_3, and S_3 replacing $(S_2 \cup S_3)$.

Proof of Inclusion relation set: $[\subset] = (((V \times 1) \cap [\cap])|((1 \times V) \cap [\cap]'))' \cap (V \times V)$

Proof of Intersection by singleton element : $[\cap^1] = ((V \times 1^\iota) \cap [\cap])|((1^\iota \times V) \cap [\cap])$

Proof of Converses relation set: $[^{-1}] = (((V \times V) \times (V \times V)) \cap ([\cap] \backslash [\cap^1])) \cup [=]^{\iota 2}$
Proof of Converses: $R^{-1} = \mathrm{dom}\,([^{-1}]|((R \times R) \cap [=]))$
Proof of 2nd projection relation set: $\pi_2 = \pi_1|[^{-1}]$

Proof of Singleton images:
We note that for each relation R we can prove the existence of a Kuratowski image of R, denoted as R^k which is: $R^k = \{<\{a\},\{a,b\}> | <a,b> \in R\}$.

Each $<\{a\},\{a,b\}>$ is $\{ \{\{a\}\}, \{\{a\},\{a,b\}\} \}$, i.e. it is $\{\{\{a\}\}, <a,b>\}$,

Clearly $R^k = (1^\iota * R) \cap \{\cap\}$.

Now a singleton image of R, denoted as R!, would be:

$R! = (R^k| ([R^{-1}]^k)^{-1}) \backslash (1^\iota \cap (R^\iota)')$