Standard Class Theory.

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Nov 4 2011

Standard class theory portrays the basic idea that all axioms of standard set theories responsible for constructing sets are derived from a kind of generalizing what is happening at the hereditarily finite level, this resulted in enabling the definition of a single axiom scheme of set construction, this scheme prove theorems of pairing, union, power, separation, replacement and infinity over sets in this theory, so it informally explain all of them. SCT is a pure class theory, no Ur-elements are encountered, just sets and proper classes.

EXPOSITION: SCT a theory in the language of FOL(=,∈)

Define: set(x) ⇔ ∃y. x ∈ y

Axioms:

I. Extensionality: ∀x. ∀y. (∀z. z ∈ x ⇔ z ∈ y) ⇒ x = y

II. Class construction schema: If ø is a formula in which x is not free, then (∃x. ∀y. y ∈ x ⇔ set(x)ø) is an axiom.

III. Set construction schema: If ø(x) is a formula in which only z1…zn occur as parameters, then

[∀z1 is HF …zn is HF. (∃x. x is a class of HF sets ∧ ø(x)) ∧ (∀x. x is a set of HF sets ∧ ø(x) ⇒ x is HF)]
⇒ (∀z1…zn are sets. (∀x. ø(x) ⇒ set(x))) is an axiom.

Where HF stand for "a Hereditarily Finite class" defined below; a class of HF sets is any class where every element of it is HF, a set of HF sets is a class of HF sets that is a set.

Subclass and *superclass* are defined in the standard manner.

A *von Neumann ordinal* is defined as a transitive class of transitive sets where every non empty subclass of it has a disjoint element of it.

A *natural number* is a Von Neumann ordinal that is either empty or a successor ordinal having every element of it either empty or a successor ordinal.

A *finite* class is a class that has a class bijection to some subclass of a natural number.

The *transitive closure* of a class is the minimal transitive superclass of that class.
A *hereditarily finite* class: is a finite class where every element of its transitive closure is a finite class.

A weaker scheme is derived if we only replace *x is a set of HF sets* with *x is a class of HF sets* in scheme III.

/ Theory definition finished.

Theorems: The first theorem is that the empty class 0 must exist; this is done from class comprehension scheme. Then we prove that 0 is HF, this can be done in few steps. Second we prove that every class that is hereditarily finite is a set; this is done by replacing the formula "x is HF" in set construction scheme. Then we prove that every class of hereditarily finite sets is a set, this is done with using the formula "$\neg$set(x)" in set construction scheme. Then we prove that every class of sets of hereditarily finite sets is a set, and this is done with the formula " x is a class of sets of HF sets $\wedge$ $\neg$set(x)". Now a Kuratowski pair of HF sets is a class of sets of HF sets, so it is a set, then by scheme II we can define class relations between any two classes of HF sets and thus decide on their bijection to a subclass of a natural number, and thus decide on their hereditarily finite status.

Now it is obvious that the pair class of any two HF sets is HF, so is the power class, union class, subclass and replacement class(with HF sets) of any HF set, thus by scheme III all those would be theorems for any sets in this theory thus proving: pairing, union, power and separation.

To prove Infinity one can easily take the formula " x is a von Neumann ordinal" and use it in scheme III, and this will readily prove the existence of an ordinal that is a set of HF sets and yet is not hereditarily finite and this can only be Omega the set of all finite von Neumann ordinals.

To prove Replacement use the formula
"$\exists$V. V={y| set(y)} $\wedge$ $\exists$F:z1$\rightarrow$ V $\wedge$ Rng(F)=x"   in scheme III.

So all axioms of ZF except Regularity and Choice are interpretable here.

A nice rather lengthy technical proof is that of proving that each natural number as defined in this theory is a true finite von Neumann ordinal in the customary sense, the following is the sketch of this proof:

Assume that k is a natural number that is not a true finite von Neumann ordinal, then k would be of the form {0,1,2,... ...,PPPk,PPk,Pk} where each P..PK refers to a predecessor of k that is a non true finite von Neumann natural number and of course it would contain all its predecessors in it including all the true finite von Neumann's. Now if the above k should exist, then this theory can prove the existence of the following class:
k*={k-1, k-2, k-3,.....},
where each k-i refers to k except all members of a true finite von Neumann ordinal i.

Now the intersectional class of k* would be provable here and it would be the class of all elements of k that are not true finite von Neumann's and this violates the condition in the definition of natural number of every subclass of it having a disjoint element of it, so k would not be a natural number, which is a contradiction.

The proof of existence of k* is done by intersecting all classes having k-1 as element of them and in which for each element of them that is a k-i for i is a natural number element of k, then the set k-(i+1) is also an element of them. Obviously the power class of k is one of those intersecting classes, and by intersection the only sets that will remain in the intersectional class are those k-i sets where i is a true finite von Neumann.