Finite Axiomatization of NF4

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A proof of finite axiomatization of NF4 which is predicative as much as possible, the main line of the proof is to prove all axioms present in Holmes book "elementary set theory with a universal set" by using ordered pairs that are just one type higher than their projections, and using parameters to reduce the type burden on formulas.

Details of the ordered pairs:

The formula *pair(p,a,b)* read as: p is an ordered pair whose first projection is a and second projection is b, abbreviated as p is a pair of a and b, is:

(p is singleton $\wedge$ $a = b \wedge a \in p$) $\vee$

(p is doubleton $\wedge$ $a \in p \wedge b \in p \wedge \varnothing \notin a \wedge \varnothing \in b$) $\vee$

(p is tripleton $\wedge$ $a \in p \wedge b \in p \wedge (\varnothing \in a <-> \varnothing \in b) \wedge a \neq b \wedge a^c \in p$) $\vee$

(p is quadruple $\wedge$ $\varnothing \in a \wedge \varnothing \notin b \wedge a \in p \wedge b \in p \wedge 1 \in p \wedge 2 \in p$) $\vee$

(p is quintuple $\wedge$ $\varnothing \in a \wedge b=1 \wedge a \in p \wedge 1 \in p \wedge 2 \in p \wedge 3 \in p \wedge 4 \in p$) $\vee$

(p is hextuple $\wedge$ $\varnothing \in a \wedge b=2 \wedge a \in p \wedge 1 \in p \wedge 2 \in p \wedge 3 \in p \wedge 4 \in p \wedge 5 \in p$)

where numbers are defined after Frege, $\varnothing$ is the empty set, a^c is the complementary set of a, and p is i_ton\ple is defined as:

$((\exists k_1..k_i\ \forall y.\ y \in p \Leftrightarrow y= k_1 \vee .. \vee y=k_i) \wedge \neg(\exists k_1..k_{i-1}\ \forall y.\ y \in p \Rightarrow y= k_1 \vee .. \vee y= k_{i-1}))$

Now the full open expansion of the formula *pair(p,a,b)* would place p at type 4. So it is a definition in four types but this would not work if we are to use it in the axioms present in Holmes book. What is needed is a definition that places p at type 2 and its projections at type 1 and of course it shouldn't exceed four types assigned to its variables. To do that we resort to the parameter approach, i.e. to parameter determined ordered pairs, so we'll modify the above to
the following:

$(p \text{ is singleton} \wedge a = b \wedge a \in p) \vee$

$(p \text{ is doubleton} \wedge a \in p \wedge b \in p \wedge a \in [\{.\emptyset.\}]^c \wedge b \in [\{.\emptyset.\}]) \vee$

$(p \text{ is tripleton} \wedge a \in p \wedge b \in p \wedge (a \in [\{.\emptyset.\}] <\!\!-\!\!> b \in [\{.\emptyset.\}]) \wedge a \neq b \wedge \exists q\, (q \in \complement \wedge a \in q \wedge b \in q)) \vee$

$(p \text{ is quadruple} \wedge a \in [\{.\emptyset.\}] \wedge b \in [\{.\emptyset.\}]^c \wedge a \in p \wedge b \in p \wedge 1 \in p \wedge 2 \in p) \vee$

$(p \text{ is quintuple} \wedge a \in [\{.\emptyset.\}] \wedge b=1 \wedge a \in p \wedge 1 \in p \wedge 2 \in p \wedge 3 \in p \wedge 4 \in p) \vee$

$(p \text{ is hextuple} \wedge a \in [\{.\emptyset.\}] \wedge b=2 \wedge a \in p \wedge 1 \in p \wedge 2 \in p \wedge 3 \in p \wedge 4 \in p \wedge 5 \in p)$

Now in order to understand what's going on, we see that the symbols for the naturals, the set $[\{.\emptyset.\}]$ and its complement, and the set $\complement$, all of those are CONSTANTS, so they can ***replace parameters***, so in this manner the symbol p can receive type level 2, while symbols a,b would receive type level 1, q would be assigned type level 2, so is the set $[\{.\emptyset.\}]$ and its complement, and $\complement$ would receive type level 3, all naturals would be assigned type level 1.

Now we come to explain what are the sets those constants stand for:

First the set denoted by the symbol $[\{.\emptyset.\}]$, this is the set of all sets containing $\emptyset$ as an element of, formally this is:

$[\{.\emptyset.\}] = \{x|\ \emptyset \in x\}$

Second is the set denoted by the symbol $\complement$, this is the set of all unordered pairs having complementary elements, formally this is:

$\complement = \{\{x,y\}|\ x=y^c\}$

The Frege numbers are defined as

$n = \{x/\ \exists y_1..y_n \forall z(z\in x \Leftrightarrow z=y_1 \vee..\vee z=y_n) \wedge \neg\exists y_1..y_{n-1} \forall z(z\in x \Rightarrow z=y_1 \vee..\vee z=y_{n-1})\}$

To make matters even clearer, and see how the type assignments would be reduced we'll use the notion *pair(p,a,b,**R,S,T,U,V,W,X,Y**)* to stand for: p is a pair of a,b after **R,S,T,U,V,W,X,Y.**

So the formula corresponding to that would be:

$(p \text{ is singleton} \wedge a = b \wedge a \in p) \vee$

$(p \text{ is doubleton} \wedge a \in p \wedge b \in p \wedge a \in \boldsymbol{R} \wedge b \in \boldsymbol{S}) \vee$

$(p \text{ is tripleton} \wedge a \in p \wedge b \in p \wedge (a \in \boldsymbol{S} <\!\!-\!\!> b \in \boldsymbol{S}) \wedge a \neq b \wedge \exists q\ (q \in \boldsymbol{T} \wedge a \in q \wedge b \in q)) \vee$

$(p \text{ is quadruple} \wedge a \in \boldsymbol{S} \wedge b \in \boldsymbol{R} \wedge a \in p \wedge b \in p \wedge \boldsymbol{U} \in p \wedge \boldsymbol{V} \in p) \vee$

$(p \text{ is quintuple} \wedge a \in \boldsymbol{S} \wedge b=\boldsymbol{U} \wedge a \in p \wedge \boldsymbol{U} \in p \wedge \boldsymbol{V} \in p \wedge \boldsymbol{W} \in p \wedge \boldsymbol{X} \in p) \vee$

$(p \text{ is hextuple} \wedge a \in \boldsymbol{S} \wedge b=\boldsymbol{V} \wedge a \in p \wedge \boldsymbol{U} \in p \wedge \boldsymbol{V} \in p \wedge \boldsymbol{W} \in p \wedge \boldsymbol{X} \in p \wedge \boldsymbol{Y} \in p)$

Of course all of what is in upper case are ***variables*** so they are not to be confused with constants. Notice the typing assignments we've spoken about earlier: clearly p would be assigned type 2, so are R,S, while a,b type 1, so are U,V,W,X,Y, only T would be assigned type 3 which is the largest type, this typing assignment appears here clearer than ever! Of course the formula is *stratified!* The formula *pair(p,a,b,R,S,T,U,V,W,X,Y)* do not always denote the pairs spoken about above, it can denote many entities some might not even be ordered pairs at all, all of this depends on what sets are substituting the parameters (upper case variables above). The idea is for those upper cases to be parameters that would be substituted by the constants we desire in order to yield the above mentioned pairs, so R would be substituted by the complement of S and S by the set $[\{.\varnothing.\}]$, T by $\mathbb{C}$, and U,V,W,X,Y by Frege numbers 1,2,3,4,5 respectively; by then for those particular instantiations the ordered pairs we've spoken about would be yielded.

Writing axioms in a parameterized manner:

To further illustrate what we want to say we'll take an example of the axiom of Cartesian products and see how we'll axiomatize it in such a way that satisfy the desired typing on pairs mentioned above.

Now suppose we are to write this axiom using non parameterized approach, i.e. using the formula *pair(p,a,b)* as:

$$\forall A,B\ \exists X\ \forall p\ (p \in X \Leftrightarrow \exists a,b\ (a \in A \wedge b \in B \wedge pair(p,a,b))$$

Then p would receive type 4 and thus X type 5 and the above axiom would not be an axiom of NF4.

Using the parameterized approach would salvage matters! We write the above axiom as:

$\forall R,S,T,U,V,W,X,Y\ \forall A,B\ \exists K\ \forall p\ (p \in K \Leftrightarrow \exists a,b\ (a \in A \wedge b \in B \wedge pair(p,a,b,R,S,T,U,V,W,X,Y)))$

Or simply: $\forall X_1..X_8\ \forall A,B\ \exists X\ \forall p\ (p \in X \Leftrightarrow \exists a,b\ (a \in A \wedge b \in B \wedge pair(p,a,b,X_1,..,X_8)))$

Clearly this would assign type 2 to p, type 1 to a,b, and the whole type assignment would not go beyond 3 actually!

Of course the plan is to substitute X_1 for the set $[\{.\varnothing.\}]^c$, X_2 for the set $[\{.\varnothing.\}]$, X_3 for the set ⊆, and X_4 through to X_8 for numbers *1* through to *5* respectively, and per those particular substitutions we get actually a theorem that is exactly the non parametric axiom of Cartesian products just stated above! This is to be referred to as: ***parametric presentation of axioms***.

The Proof that NF can be axiomatized finitely in 4 types.

Axioms:
P1 of Haiplerin
Singletons
Set Union
Frege numbers 1 though to 5 exist,
⊆ exists.
$[\{.\varnothing.\}]$ exists.

All of those can be written using four types only, those are written directly without any need for any parameters.

Axioms:
Cartesian products
Domains
Converses
Relative products
Diagonal
Singleton images of relations
Projections
Inclusion

All to be presented parametrically in the same way Cartesian products above was presented parametrically.
QED

Observation: We note that the axiom "$[\{.\emptyset.\}]$ exists" can be easily presented parametrically in order to make it axiomatizable in NF3. Also if we add "ℂ exists" to the axioms of NF3, i.e. we get NF3+ℂ, then we can write all axioms from Hailpern P1 through to diagonal parametrically in *three* types! Thus:

NF= NF3+ℂ+singleton images+projections+Inclusion.

Observation: We note that all axioms except union adhere to predicative standards, so NFP is finitely axiomantizable in four types without any need for infinity.

Another proof based on a simpler ordered pair:

The known pair $\{\{x,0,1\},\{x,2,3\},\{y,4,5\},\{y,6,7\} \mid x \in A, y \in B\}$
which stands for: the ordered pair of A and B, can also be used to prove NF4.

Add axioms stipulating existence of the following sets:

$K=\{\{x,0,1\} \mid x=x\}$
$L=\{\{x,2,3\} \mid x=x\}$
$M=\{\{x,4,5\} \mid x=x\}$
$N=\{\{x,6,7\} \mid x=x\}$
$G=\{\{a,b\} \mid b \in K \cup L \land \exists x.\ x \in a \land x \in b \land x \neq 0 \land .. \land x \neq 3\}$
$H=\{\{a,b\} \mid b \in M \cup N \land \exists x.\ x \in a \land x \in b \land x \neq 4 \land .. \land x \neq 7\}$
$SING=\{\{x\} \mid x=x\}$
$TRIO=\{\{a,b,c\} \mid a \neq b \neq c\}$
$SUB=\{\{\{x\},y\} \mid x \in y\}$
$X.Y= \{x \mid x \in X \land x \in Y\}$

Definition: p is an ordered pair iff $p.K \neq \emptyset \land p.L \neq \emptyset \land p.M \neq \emptyset \land p.N \neq \emptyset \land$

$\forall z \in SING\ (\exists m \in p.K\ \exists g \in G\ (m \in g \land z \in g) \Rightarrow \exists n \in p.L\ \exists s \in SUB\ (n \in s \land z \in s))$
$\land$
$\forall z \in SING\ (\exists m \in p.L\ \exists g \in G\ (m \in g \land z \in g) \Rightarrow \exists n \in p.K\ \exists s \in SUB\ (n \in s \land z \in s))$
$\land$
$\forall z \in SING\ (\exists m \in p.N\ \exists h \in H\ (m \in h \land z \in h) \Rightarrow \exists n \in p.M\ \exists s \in SUB\ (n \in s \land z \in s))$
$\land$
$\forall z \in SING\ (\exists m \in p.M\ \exists h \in H\ (m \in h \land z \in h) \Rightarrow \exists n \in p.L\ \exists s \in SUB\ (n \in s \land z \in s))$

Now we come to definitions of 1st and 2n projections of an ordered pair:

first(a,p) ⇔ p is an ordered pair ∧
[∀x. x ∈ k U L ⇒ [(∃g. g ∈ G ∧ a ∈ g ∧ x ∈ g) ⇔ x ∈ p ∧ x ∈ TRIO] ∧

second(b,p) ⇔ p is an ordered pair ∧
[∀x. x ∈ M U N ⇒ [(∃h. h ∈ H ∧ b ∈ h ∧ x ∈ h) ⇔ x ∈ p ∧ x ∈ TRIO]

Both definitions place each projection at type level 1, the pair itself at type level 2 and the formula doesn't exceed type level 3 which is assigned to parameters only.

That's it! since we have those definitions then it is *STRAIGHT FORWARDS* to prove NF4, we only follow the parameterized approach mentioned earlier, all axioms in Holmes book on finite axiomatization of NF can be written in a parameterized manner consuming four types only.
QED

A proof of NFU4 and NFUP4:

The above proofs only pertain to NF, since both kinds of pairs require Extensionality. Therefore we cannot utilize them to prove NFU4 and its fragments.

The following is a proof of NF4 using the familiar Kuratowski pairs. Of course this proof can be used also as a proof of NF4, since extensionality requires 2 types only.

PROOF: We note that all axioms in Holmes book except axioms of inclusion, projections and singleton images, are easily written in no more than four types and in a predicative manner (except union which is impredicative).

Add the following axioms:

∃x. x={y| ∃z ∀m. m ∈ y ⇔ m=z}
∃x. x={{a,b}| ∃y. y ∈ a & y ∈ b}

Let’s call the first set 1 and the second set [∩]

The axiom of Inclusion (Haiplerin axiom P9) would be written as

$\exists x.\ \forall p.\ p \in x \Leftrightarrow \exists ab{:}\ p=(a,b) \wedge a \in 1 \wedge \{a,b\} \in\ [\cap]$

Call this set as: $[\subset]$

The axiom of Singleton images would be written as:

$$\forall A\ \exists x\ \forall p.\ p \in x \Leftrightarrow \exists z,a,b{:}\ z \in A \cap VxV \wedge p=(a,b) \wedge a \in 1 \wedge b \in 1 \wedge$$
$$(\forall u.\ u \in z \Rightarrow \{a,u\} \in [\cap]) \wedge (\exists! w.\ w \in z \wedge \{b,w\} \in [\cap])$$

For axioms of projections we need to stipulate existence of the following deep intersection set:

$[\cap]' = \{\{a,b\} | a \in 1 \wedge\ \forall m \in a\ (\exists! n.\ n \in b \wedge \{m,n\} \in [\cap])\}$

Definition: $1'' = \{\{\{x\}\} |\ x=x\}$

Now the first projection relation set is definable as: $1''x(VxV) \cap [\subset]$

The second projection relation set is to be axiomatized as:

$\exists x\ \forall p.\ p \in x \Leftrightarrow \exists a,b{:}\ p=(a,b) \wedge a \in 1'' \wedge b \in VxV \wedge \{a,b\} \in [\cap]'$

Of course it is clear that all constants are to be treated as Parameters!

of course domains and diagonal are redundant axioms if we are using the Kuratowski pairs.

So all axioms in Holmes book on finite axiomatization of NFU are written in parameterized manner in four types and predicatively (except union) so!! without any need for full extensionality nor for infinity! So NFUP4 is proved here as well.
QED

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