Disguised Set Theory "DST"

By

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EXPOSITION

DST is the collection of all sentences in first order logic with equality "=" and membership"ϵ" entailed from the following non logical axioms:

Extensionality: $\forall x.\ \forall y.\ (\forall z.\ z \epsilon x \Leftrightarrow z \epsilon y) \Rightarrow x = y.$

Define ($\subset$): $x \subset y \Leftrightarrow \forall z \epsilon x.\ (z \epsilon y)$

Define: $T(x) \Leftrightarrow \forall y \epsilon x.\ (y \subset x)$

Transitive Closure: $\forall x.\ \exists y.\ T(y) \wedge x \subset y \wedge \forall z.\ (T(z) \wedge x \subset z) \Rightarrow y \subset z.$

Define (TC): $TC(x) = y \Leftrightarrow T(y) \wedge x \subset y \wedge \forall z.\ (T(z) \wedge x \subset z) \Rightarrow y \subset z$

Define ($\in$): $x \in y \Leftrightarrow x \epsilon y \wedge \neg y \epsilon TC(x)$

Disguised Comprehension: if ø is a formula that only uses predicates = and $\in$, and in which x do not occur free, then $(\exists x.\ \forall y.\ y \epsilon x \Leftrightarrow ø)$ is an axiom.

Define ({|}): $x = \{y|\ ø\} \Leftrightarrow (\forall y.\ y \epsilon x \Leftrightarrow ø)$

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This theory definitely gives the appearance of an inconsistent theory, but it is not easy to find such an inconsistency if there is any.

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M. Randall Holmes has suggested that an axiom of Induction over transitive closures is necessary.

Induction: if ø is a formula, then
$(\forall x.\ [(\forall y \epsilon x.\ ø(y)) \wedge (\forall yz.\ z \epsilon y \wedge ø(y) \Rightarrow ø(z))] \Rightarrow (\forall y \epsilon TC(x).\ ø(y)))$ is an axiom.

Define (acyclic): x is acyclic $\Leftrightarrow \neg x \epsilon TC(x)$

Define (hereditarily acyclic): x is hereditarily acyclic $\Leftrightarrow (\forall y \epsilon TC(x).$ y is acyclic$)$

This theory interprets bounded ZF-Power-Infinity over hereditarily acyclic sets.

Another axiom that I was considering to add to this theory is the axiom of Filtering, whereby we can filter out unwanted non-hereditarily acyclic sets.

Filtering: $\exists x.$ [$\forall y.$ y is hereditarily acyclic ($y \in x$)] $\wedge$ [$\forall zu.$ ($z \in x \wedge u$ is hereditarily acyclic $\wedge$ ($\forall m.\ m \in z \Leftrightarrow m \in u$)) $\Rightarrow z = u$]

Define(filter): x is a filter $\Leftrightarrow$ [$\forall y.$ y is hereditarily acyclic ($y \in x$)] $\wedge$ [$\forall zu.$ ($z \in x$ $\wedge$ u is hereditarily acyclic $\wedge$ ($\forall m.\ m \in z \Leftrightarrow m \in u$)) $\Rightarrow z = u$]

With the use of these filters one can easily prove both power and Infinity and so the theory would interpret the whole of bounded ZF over hereditarily acyclic sets.

Versions of ZF with less boundedness can also be interpreted over hereditarily acyclic sets, thus stronger fragments of ZF are interpretable here.

The consistency of this theory relative to known theories is still not established yet; it might probably turn to be inconsistent.

Of course the theory has a universal set! So many kinds of big sets do exist in this theory. M. Randall Holmes has discovered a descending hierarchy of big sets of the form $V\_i+1 = \{x | x \in V\_i\}$ for each concrete natural number i where V_0 stands for V (the set of all sets).

It would be interesting to see what this theory can tell us about the big sets in it.

Also it would be nice to see what strength of known theories this theory can interpret.

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DST alone is shown to prove Infinity, see: http://zaljohar.tripod.com/dstinf.pdf

DST with an a special axiom of Extensionality is shown to prove ZF, see: http://zaljohar.tripod.com/interpreting_zfc_in_dst.pdf